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On tree–level higher order gravitational couplings in superstring theory

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We consider the scattering amplitudes of five and six gravitons at tree–level in superstring theory. Their power series expansions in the Regge slope α' are analyzed through the order α'^8 showing some interesting constraints on higher order gravitational couplings in the effective superstring action like the absence of R^5 terms. Furthermore, some transcendentality constraints on the coefficients of the non–vanishing couplings are observed: the absence of zeta values of even weight through the order α'^8 like the absence of $\zeta(2)\zeta(3)R^6$ terms. Our analysis is valid for any superstring background in any space–time dimension, which allows for a conformal field theory description.

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Superstring theories contain a massless spin–two state identified as a graviton. Its interactions are studied by graviton scattering amplitudes. Due to the extended nature of strings the latter are generically non-trivial functions on the string tension α'. In the effective field theory (FT) description this α'-dependence gives rise to a series of infinite many higher order gravitational couplings governed by positive integer powers in α'. The classical Einstein–Hilbert term is reproduced in the zero–slope limit $\alpha'\to 0$. The modification of the Einstein equations through the order α'^3 has been derived by studying tree–level four graviton scattering amplitudes [1–4] or alternatively by computing four–loop β–functions of the underlying σ–model [5, 6]. Up to this order the effective (tree–level) superstring action focusing on pure gravitational bosonic terms reads in D space–time dimensions

$$\mathcal{L}_{\rm tree}=\frac{1}{2\kappa^2}\ R+\frac{\alpha'^3}{2^9\ 4!\ \kappa^2}\ \zeta(3)\ t_8t_8\ R^4\ , \qquad (1)$$

with the gravitational coupling constant κ in D dimensions, the Riemann scalar R, the Riemann tensor $R_{\mu\nu\rho\sigma}$ and the tensor t_8 defined in equation (4.A.21) of [7]:

$$\begin{aligned} t_8t_8R^4 &\equiv t_8^{\mu_1\mu_2\ldots\mu_8}t_8^{\nu_1\nu_2\ldots\nu_8} \qquad (2)\\ &\times R_{\mu_1\mu_2\nu_1\nu_2}R_{\mu_3\mu_4\nu_3\nu_4}R_{\mu_5\mu_5\nu_5\nu_6}R_{\mu_7\mu_8\nu_7\nu_8}\ .\end{aligned}$$

If the indices are restricted to $D=4$ the combination $t_8t_8R^4$ becomes the Bel–Robinson tensor. The absence of R^2 and R^3 terms in superstring theory is shown in [8].

In $D=4$ the result simply follows by expanding the (only independent and non–vanishing) four–graviton amplitude

$$\mathcal{M}(1^-,2^-,3^+,4^+)=\left(\frac{\kappa}{2}\right)^2\frac{\langle 12\rangle^8[12]}{N(4)\langle 34\rangle}\frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})} \qquad (3)$$

through the order α'^3. The superscripts $\pm$ denote the helicities of the corresponding gravitons. Above, we have introduced the kinematic invariants $s_{ij}=2\alpha' k_ik_j$ involving the external (on–shell) momenta k_i and the Euler Beta function B encoding the α'–dependence of the full string amplitude. Furthermore, $\langle ij\rangle, [ij]$ are spinor products (see e.g. [9, 10]), $s_{ij}=\alpha'\ \{i,j\}:=\alpha'\langle ij\rangle[ji]$, and $N(n)=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\langle ij\rangle$ [11].

In Eq. (1) further α'–corrections $\mathcal{L}'_{\rm tree}$ arise from terms with higher powers in the Riemann tensor $R_{\mu\nu\rho\sigma}$ supplemented by covariant derivatives D. In the following these terms are collectively denoted by $t_{m,n}D^mR^n$, with some tensor $t_{m,n}$ contracting D and the Riemann tensor R. Generically, the set of these additional interactions may be summarized in the series

$$\begin{aligned}\mathcal{L}'_{\rm tree}=\kappa^{-2}\sum_{n\geq 4}^{\infty}\sum_{m=0}^{\infty}\alpha'^{n-1+m}&\sideset{}{'}\sum_{\substack{i_r\in\mathbf{N},i_1>1\\ i_1+\ldots+i_d=n-1+m}}\zeta(i_1,\ldots,i_d)\\ &\times c_{m,n,\boldsymbol{i}}\ \ t^{\boldsymbol{i}}_{m,n}D^{2m}R^n\ , \qquad (4)\end{aligned}$$

with multi–zeta values (MZVs)

$$\zeta(i_1,\ldots,i_d)=\sum_{n_1>\ldots>n_d>0}\prod_{r=1}^{d}n_r^{-i_r},\quad i_r\in\mathbf{N},\ i_1>1$$

of transcendentality degree $\sum_{r=1}^d i_r{=}n-1+m$ and depth d supplemented by some rational coefficients $c_{m,n,\boldsymbol{i}}$. The prime at the sum (4) means, that the latter runs only over a basis of independent MZVs of weight $n-1+m$ [12–15]. For a recent account on MZVs see Ref. [16].

The terms in the sum (4) are probed by computing scattering amplitudes of n gravitons and analyzing their power series in α' [17]. For $n{=}4$ it is straightforward to extract the relevant information, since in this case each term in the α'–expansion of (3) directly translates into a term in the effective action (4). Moreover, the MZV coefficients of the latter are simply products of Riemann zeta functions $\zeta(i_1)$ of odd degree i_1 as a consequence of:

$$\frac{B(s_{12},s_{14})}{B(-s_{12},-s_{14})}=-e^{-2\sum\limits_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}(s_{12}^{2n+1}+s_{13}^{2n+1}+s_{14}^{2n+1})}\ . \qquad (5)$$

In this note we discuss the consequences of (on–shell) superstring scattering of five and six gravitons to the correction terms (4). We find, that some of the coefficients $c_{m,n,\boldsymbol{i}}$ are vanishing and the MZVs of the non–vanishing terms follow some specific pattern.

The string world–sheet describing the tree–level string S–matrix of N gravitons is described by a complex sphere with N (integrated) insertions z_i of graviton vertex operators $V_G(\epsilon,\overline{z}_i,z_i)$:

$$\begin{aligned}\mathcal{M}(1,\ldots,N)&=V_{CKG}^{-1} \qquad (6)\\ &\times\left(\prod_{j=1}^{N}\int_{\mathbf{C}}d^2z_j\right)\langle V_G(\epsilon_1,\overline{z}_1,z_1)\ldots V_G(\epsilon_N,\overline{z}_N,z_N)\rangle.\end{aligned}$$

Here, the factor V_{CKG} accounts for the volume of the conformal Killing group. One of the key properties of graviton amplitudes in string theory is that at tree–level they can be expressed as sum over squares of (color ordered) gauge amplitudes in the left– and right–moving sectors. This map, known as Kawai–Lewellen–Tye (KLT) relations [18], gives a relation between a closed string tree–level amplitude on the sphere and a sum of squares of (partial ordered) open string disk amplitudes. On the string world–sheet of the sphere describing the N–graviton amplitude (6) the KLT relations are a consequence of decoupling holomorphic and anti–holomorphic sectors by splitting the complex sphere integration over the coordinates $\overline{z}, z \in \mathbf{C}$ into two real ones $\eta, \xi \in \mathbf{R}$ describing products of two open string disk amplitudes. At the level of degrees of freedom this is anticipated from the fact, that the graviton vertex operator V_G splits into a product of non–interacting open string states describing the vertex operator of massless vectors V_g, i.e.: $V_G(\epsilon, \overline{z}, z) \simeq V_g(\overline{\epsilon}, \eta)\ \times V_g(\epsilon, \xi)$, subject to the decomposition of polarization tensors $\epsilon_{\mu\nu}{=}\overline{\epsilon}_\mu \otimes \epsilon_\nu$. In the graviton amplitude (6) the latter comprises into the linearized Riemann tensor $R_{\mu\nu\rho\sigma}{=}\kappa\ k_{[\mu} k_{[\rho}\ \overline{\epsilon}_{\nu]} \otimes \epsilon_{\sigma]}$.

For the two cases $N = 5, 6$, which we shall consider in the sequel, we have the following relations (with $\mathcal{M}(1,\ldots,N){=}(\frac{\kappa}{2})^{N-2} M(1\ldots N)$) [18]:

$$M(12345) = (2\alpha'\pi)^{-2}\ \sin(\pi s_{12})\ \sin(\pi s_{34})\ \bar{A}(12345)\ A(21435) + (\overline{23}), \tag{7}$$

$$M(123456) = (2\alpha'\pi)^{-3} \sin(\pi s_{12}) \sin(\pi s_{45}) \bar{A}(123456)\, \{\sin(\pi s_{35}) A(215346) + \sin[\pi(s_{34}+s_{35})] A(215436)\} + (\overline{234}). \tag{8}$$

The KLT relations (7,8) hold for any superstring background and are insensitive to the compactification details or the amount of supersymmetries. The gluon amplitudes $A(1\ldots N)$ are the color–ordered subamplitudes of the underlying gauge theory. Hence, in superstring theory the tree–level computation of graviton amplitudes boils down to considering squares of tree–level gauge amplitudes A. For this sector explicit computations have been performed and results are accessible for the cases N=4 [7, 19], for N=5 [20–23], for N=6 [21–24] and N=7 [25]. Moreover, through the order α'^2 the full N–gluon MHV–amplitude is given in [22, 23], while the order α'^3 has been constructed in [26]. Based on all these results and the relations (7,8) the five– and six–graviton amplitudes can be derived.

The result for the five–gluon subamplitude can be given for any space–time dimension D in the form [27]

$$A(12345) = C_1\ A_{YM}(12345) + C_2\ A_{F^4}(12345)\ , \tag{9}$$

with the kinematical factors encoding the pure YM part $A_{YM}(12345)$ and the genuine string part $A_{F^4}(12345)$

$$\begin{aligned} A_{F^4}(12345) = (2\alpha')^2\ \{\ & K(\epsilon_1,\epsilon_2,\epsilon_3,k_3,\epsilon_4,k_4,\epsilon_5,k_5) + (k_1k_2)^{-1}\ [\ (\epsilon_1\epsilon_2)\ K(k_1,k_2,\epsilon_3,k_3,\epsilon_4,k_4,\epsilon_5,k_5) \\ & + (\epsilon_1 k_2)\ K(\epsilon_2,k_1+k_2,\epsilon_3,k_3,\epsilon_4,k_4,\epsilon_5,k_5) - (\epsilon_2 k_1)\ K(\epsilon_1,k_1+k_2,\epsilon_3,k_3,\epsilon_4,k_4,\epsilon_5,k_5)\ ] \\ & + \text{cyclic permutations}\ \}\ ,\ \text{with}:\ K(\epsilon_1,k_1,\epsilon_2,k_2,\epsilon_3,k_3,\epsilon_4,k_4) = t_8^{\mu_1\ldots\mu_8} \epsilon_1^{\mu_1} k_1^{\mu_2} \ldots \epsilon_4^{\mu_7} k_4^{\mu_8}. \end{aligned} \tag{10}$$

Furthermore, we have the two (basis) functions

$$C_1 = s_2 s_5\ f_1 + (s_2 s_3 + s_4 s_5)\ f_2 \quad , \quad C_2 = f_2\ , \tag{11}$$

defined by the two hypergeometric functions ${}_3F_2$:

$$\begin{aligned} f_1 &= \int_0^1 dx \int_0^1 dy\ x^{s_2-1} y^{s_5-1} (1-x)^{s_3} (1-y)^{s_4} (1-xy)^{s_{35}}\ , \\ f_2 &= \int_0^1 dx \int_0^1 dy\ x^{s_2} y^{s_5} (1-x)^{s_3} (1-y)^{s_4} (1-xy)^{s_{35}-1}\ . \end{aligned} \tag{12}$$

In D=4 spinor notation the expressions in (9) boil down to the maximally helicity violating YM five–point partial amplitude [28, 29]

$$A_{YM}(1^-2^-3^+4^+5^+) = i\ \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\ , \tag{13}$$

and to the term

$$\begin{aligned} A_{F^4}(1^-2^-3^+4^+5^+) = \alpha'^2\ (\ & \langle 12\rangle[23]\langle 34\rangle[41] \\ & -\ \{3,4\}\{4,5\} - \{1,2\}\{1,5\}\ ) \\ & \times A_{YM}(1^-2^-3^+4^+5^+) \end{aligned} \tag{14}$$

describing the leading string correction to the YM amplitude (13). In the effective action (14) gives rise to the contact term $t_8 F^4 \equiv t_8^{\mu_1\nu_1\ldots\mu_4\nu_4} F_{\mu_1\nu_1} F_{\mu_2\nu_2} F_{\mu_3\nu_3} F_{\mu_4\nu_4}$ with the field strength $F_{\mu\nu}$.

Using in (7) the expression (9) yields the full five–graviton amplitude. Its α'–expansion is determined by expanding the basis (12), which is achieved with [30]. In the D=4 FT–limit, with $f_1 \overset{\alpha'\to 0}{\to} \frac{1}{s_2 s_5}$ and $f_2 \overset{\alpha'\to 0}{\to} 0$, the only independent five graviton amplitude reduces to [11]

$$M(1^-2^-3^+4^+5^+)\big|_{\alpha'^0} = -4i\ \frac{\langle 12\rangle^8}{N(5)}\ \epsilon(1,2,3,4), \tag{15}$$

with $4i\epsilon(1,2,3,4) = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$. Through the order α'^8 we find for any dimension D:

$$\begin{aligned}
&M(12345)|_{\alpha'^2} = 0, \qquad M(12345)|_{\alpha'^4} = 0, \qquad M(12345)|_{\alpha'^n} = \zeta(n)\ m(12345)|_{\zeta(n)} + (\overline{23})\ , \quad n = 3,5,7\ ,\\
&M(12345)|_{\alpha'^6} = \zeta(3)^2\ m(12345)|_{\zeta(3)^2} + \zeta(3)^2\ \hat s_{12}\hat s_{34}\left(\bar C_1 \bar A_{YM} + \bar C_2 \bar A_{F^4}\right)|_{\zeta(3)}\left(C_1 A_{YM} + C_2 A_{F^4}\right)|_{\zeta(3)} + (\overline{23}),\\
&M(12345)|_{\alpha'^8} = \zeta(3)\zeta(5)\ m(12345)|_{\zeta(3)\zeta(5)} + \zeta(3)\zeta(5)\ \hat s_{12}\hat s_{34}\ \left[\left(\bar C_1 \bar A_{YM} + \bar C_2 \bar A_{F^4}\right)|_{\zeta(3)}\left(C_1 A_{YM} + C_2 A_{F^4}\right)|_{\zeta(5)}\right.\\
&\qquad\qquad \left. +\ \left(\bar C_1 \bar A_{YM} + \bar C_2 \bar A_{F^4}\right)|_{\zeta(5)}\ \left(C_1 A_{YM} + C_2 A_{F^4}\right)|_{\zeta(3)}\right] + (\overline{23})\ . \qquad (16)
\end{aligned}$$

Above we have introduced ($\hat s_{ij} = (2\alpha')^{-1} s_{ij}$):

$$\begin{aligned} m(12345) = \hat s_{12}\hat s_{34}\left[\left(\bar C_1 \bar A_{YM} + \bar C_2 \bar A_{F^4}\right) A_{YM}\right.\\ \left. +\ \bar A_{YM}\left(C_1 A_{YM} + C_2 A_{F^4}\right)\right]\ . \qquad (17)\end{aligned}$$

Let us now discuss the implications of the results (16). In [31–34] the four–graviton amplitude has been analyzed resulting in a series of higher order terms $t_8 t_8 D^{2m} R^4$, which enter Eq. (4) with zeta functions shown in the first column of Table I. The (on-shell) four–graviton amplitude does not contain the momentum terms corresponding to D^2R^4 [35]. The only possible Feynman diagrams contributing at the order $\alpha'^{3+m}, m \geq 0$ to the five–graviton amplitude are displayed in Fig. 1. For $m=0$ the above diagrams simply reproduce the α'^3–order of the five–graviton amplitude involving the R^4 term.

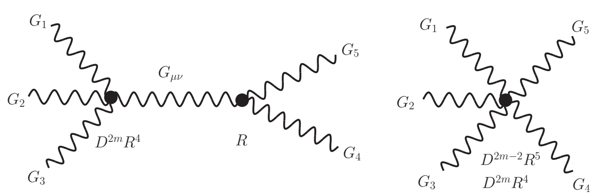

FIG. 1: *Diagrams contributing at the order α'^{3+m} to the five–graviton amplitude.*

Any D^2R^4 term can always be rewritten as a sum of R^5 terms due to the relation $D^2R \simeq R^2$. Hence, for $m=1$ only the 5–vertex (right diagram in Fig. 1) stemming from an R^5 term may contribute at the order α'^4 to the five–graviton amplitude. Since the latter vanishes at this order, cf. (16), we conclude that an R^5 term does not exist. At higher orders $\alpha'^{3+m}, m \geq 2$ both $D^{2m-2}R^5$ and $D^{2m}R^4$ terms may contribute in Fig. 1. The α'–expansion of the five–graviton amplitude does not show $\zeta(2)\zeta(3)$–terms at α'^5, $\zeta(6)$–terms at α'^6, $\zeta(3)\zeta(4)$ nor $\zeta(2)\zeta(5)$–terms at α'^7, and not any $\zeta(8)$, $\zeta(2)\zeta(3)^2$ and $\zeta(5,3)$–terms at α'^8, cf. Eqs. (16). Since those terms cannot be generated by any reducible diagram with five external gravitons we conclude the absence of contact interactions with coefficients $\alpha'^5\zeta(2)\zeta(3), \alpha'^6\zeta(6),\ \alpha'^7\zeta(3)\zeta(4), \alpha'^7\zeta(2)\zeta(5)$, and $\alpha'^8\zeta(8), \alpha'^8\zeta(2)\zeta(3)^2, \alpha'^8\zeta(5,3)$, respectively, cf. second column of Table I. On the other hand, the non–vanishing terms (16), which are proportional to $\alpha'^5\zeta(5)$, $\alpha'^6\zeta(3)^2$, $\alpha'^7\zeta(7)$, and $\alpha'^8\zeta(3)\zeta(5)$, respectively, account for the two diagrams in Fig. 1. After subtracting the contribution of the left diagram the remaining terms give rise to combinations of the interactions D^4R^4, D^2R^5 at α'^5, combinations of D^6R^4, D^4R^5 at α'^6, D^8R^4, D^6R^5 at α'^7 and $D^{10}R^4, D^8R^5$ terms at α'^8, respectively.

	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$
$\alpha'^3\ \zeta(3)$	R^4				
$\alpha'^4\ \cancel{\zeta(4)}$	$\cancel{D^2R^4}$	$\cancel{R^5}$			
$\alpha'^5\ \zeta(5)$	D^4R^4	D^2R^5	R^6		
$\alpha'^5\ \cancel{\zeta(2)\zeta(3)}$	$\cancel{D^4R^4}$	$\cancel{D^2R^5}$	$\cancel{R^6}$		
$\alpha'^6\ \zeta(3)^2$	D^6R^4	D^4R^5	D^2R^6	R^7 ?	
$\alpha'^6\ \zeta(6)$	$\cancel{D^6R^4}$	$\cancel{D^4R^5}$	$\cancel{D^2R^6}$	R^7 ?	
$\alpha'^7\ \zeta(7)$	D^8R^4	D^6R^5	D^4R^6	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(3)\zeta(4)$	$\cancel{D^8R^4}$	$\cancel{D^6R^5}$	$\cancel{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^7\ \zeta(2)\zeta(5)$	$\cancel{D^8R^4}$	$\cancel{D^6R^5}$	$\cancel{D^4R^6}$	D^2R^7 ?	R^8 ?
$\alpha'^8\ \zeta(3)\zeta(5)$	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(8)$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(2)\zeta(3)^2$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?
$\alpha'^8\ \zeta(5,3)$	$\cancel{D^{10}R^4}$	$\cancel{D^8R^5}$	$\cancel{D^6R^6}$	D^4R^7 ?	D^2R^8 ?

TABLE I. *Tree–level higher order gravitational couplings and their corresponding zeta value coefficients probed by the N–graviton superstring amplitude. Vanishing terms are crossed out. Those terms, which have not yet been probed by the relevant N–graviton amplitude, are marked by a question mark.*

Next, we consider the scattering of six gravitons. The result for the six–gluon subamplitude is given for any space–time dimension D in [21], while in D=4 spinor notation in [22–24]. Using these expressions in (8) yields the six–graviton amplitude. The α'–expansion of this amplitude gives vanishing results at the orders α'^2 and α'^4:

$$M(123456)|_{\alpha'^2} = 0,\ \ M(123456)|_{\alpha'^4} = 0. \qquad (18)$$

The order α'^3 is proportional to $\zeta(3)$ and describes diagrams involving vertices from the R^4 coupling. Moreover, through the order α'^8 we find the following properties:

$$\begin{aligned}
&M(123456)|_{\zeta(2)\zeta(3)\alpha'^5} = 0,\ \ M(123456)|_{\zeta(6)\alpha'^6} = 0,\\
&M(123456)|_{\zeta(2)\zeta(5)\alpha'^7} = 0,\ \ M(123456)|_{\zeta(3)\zeta(4)\alpha'^7} = 0,\\
&M(123456)|_{\zeta(8)\alpha'^8} = 0, \qquad M(123456)|_{\zeta(2)\zeta(3)^2\alpha'^8} = 0,\\
&M(123456)|_{\zeta(5,3)\alpha'^8} = 0\ . \qquad (19)
\end{aligned}$$

Together with the previous results the findings (19) restrict the contact interactions at α'^5 to be of the form $\zeta(5)\{D^4R^4, D^2R^5, R^6\}$, but forbids any $\zeta(2)\zeta(3)$ contact terms at this order. Similarly, contact interaction at α'^6 may assume the form $\zeta(3)^2\{D^6R^4, D^4R^5, D^2R^6\}$, but no

interactions with $\zeta(6)$–factors are possible. At this order also a reducible diagram describing the exchange of a graviton between two four–vertices of $\zeta(3)R^4$ contributes. At the order α'^7 only contact terms of the form $\zeta(7)\{D^8R^4, D^6R^5, D^4R^6\}$ may appear. However, no contact interactions with $\zeta(3)\zeta(4)$ nor $\zeta(2)\zeta(5)$–factors exist at this order in α'. Eventually, at α'^8 only contact terms of the form $\zeta(3)\zeta(5)\{D^{10}R^4, D^8R^5, D^6R^6\}$ may appear. However, no contact interactions with $\zeta(8), \zeta(2)\zeta(3)^2$ nor $\zeta(5,3)$–factors exist at α'^8. What remains to be checked is how for a given order in α' the set of contact interactions belonging to one row of Table I can be expressed by a minimal basis of terms. The latter may allow to reduce the number of Riemann tensors R by converting them into derivatives D^2, cf. the comment [12].

The results presented here and summarized in Table I hold for any type I or II superstring compactification in D space–time dimensions with eight or more supercharges and suggest that higher order gravitational couplings (4) obey some refined transcendentality properties: at each order in α' only Riemann zeta functions of odd weight or products thereof appear. While for the first column this is obvious to all orders in α' as a result of the relation (5), for the second and third column we have checked this statement up to the order α'^8. Hence, for $n \leq 6$ and up to order α'^8 the sum (4) runs only over basis elements comprised by MZVs of odd weights [15]. The absence of the MZV $\zeta(5,3)$ at the order α'^8 fits into this criterium, since it may be written as $\zeta(5,3) = -\frac{5}{2}\zeta(6,2) - \frac{21}{25}\zeta(2)^4 + 5\zeta(3)\zeta(5)$.

For $D=10$ type IIB superstring theory our findings together with the one–loop results [36] restrict the ring of possible modular forms describing the perturbative and non–perturbative completion of the higher order terms.

The structure of the α'–expansion takes over to amplitudes with some of the external gravitons replaced by some other member of the supergravity multiplet. E.g. in $D=4$, $\mathcal{N}=8$ the Fock space decomposition $\left|\begin{smallmatrix}\mathcal{N}=8\\ SUGRA\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}\mathcal{N}=4\\ SYM\end{smallmatrix}\right\rangle \otimes \left|\begin{smallmatrix}\mathcal{N}=4\\ SYM\end{smallmatrix}\right\rangle$ of the 256 states of the $\mathcal{N}=8$ supergravity multiplet selects the corresponding states in the gauge sectors, i.e. open string sectors. In these sectors different amplitudes can be related to all orders in α' by using supersymmetric Ward identities [25]. After applying the KLT relations the organization of the α'–expansion stays the same as in the graviton case and the constraints for $D^{2m}R^n$ terms take over to their supersymmetric variants.

Our amplitude results, summarized in Table I, have impact on the recently discussed finiteness of $\mathcal{N}=8$ supergravity in $D=4$. Counterterms invariant under $\mathcal{N}=8$ supergravity have an unique kinematic structure and the tree–level string amplitudes provide candidates for them, which are compatible with SUSY Ward identities and locality. The absence or restriction on higher order gravitational terms at the order α'^l together with their symmetries constrain the appearance of possible counter terms available at l–loop, see [37] for a review and references therein.

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[14] The set of integral linear combinations of MZVs is a ring, since the product of any two values can be expressed by a (positive) integer linear combination of the other MZVs [13], e.g.: $\zeta(m)\zeta(n) = \zeta(m,n)+\zeta(n,m)+\zeta(m+n)$ (quasi–shuffle or stuffle relation).

[15] There are many relations over **Q** among MZVs, e.g.

$\zeta(4,1){=}2\zeta(5)-\zeta(2)\zeta(3)$. For a given weight $w\in\mathbf{N}$ the dimension d_w of the space spanned by MZVs of weight w is given by $d_w=d_{w-2}+d_{w-3}$ ($d_1=0,d_2=d_3=d_4=1,d_5=2,d_6=2,d_7=3,d_8=4,d_9=5,d_{10}=7,\ldots$) [13] and can be constructed by the following basis: for $w=2$ by $\zeta(2)$, for $w=3$ by $\zeta(3)$, for $w=4$ by $\zeta(2)^2$, for $w=5$ by $\zeta(5),\zeta(2)\zeta(3)$, for $w=6$ by $\zeta(2)^3,\zeta(3)^2$, for $w=7$ by $\zeta(7),\zeta(2)\zeta(5),\zeta(3)\zeta(2)^2$, for $w{=}8$ by $\zeta(2)^4,\zeta(2)\zeta(3)^2,\zeta(3)\zeta(5),\zeta(5,3)$, for $w=9$ by $\zeta(9),\zeta(7)\zeta(2),\zeta(5)\zeta(2)^2,\zeta(3)^3,\zeta(3)\zeta(2)^3$, for $w=10$ by $\zeta(7,3)$,$\zeta(5,3)\zeta(2),\zeta(7)\zeta(3),\zeta(5)^2,\zeta(5)\zeta(3)\zeta(2),\zeta(3)^2\zeta(2)^2$, $\zeta(2)^5$, etc.

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